Last Class: $\Phi : G \to H$ homomorphism

1. $\ker \Phi$ is a normal subgroup of G

$\Rightarrow$ can form factor group $G/\ker\Phi$

2. First Isomorphism Theorem:

$G/\ker\Phi$ is isomorphic to $\Phi(G)$

Examples:

1. $\Phi : A \in GL(2,\mathbb{R}) \longrightarrow \det A \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$

Φ is a group hom. with $\ker\Phi = \{A,\ \det(A) = 1\}$
$= SL(2,\mathbb{R})$

$\Rightarrow GL(2,\mathbb{R})/SL(2,\mathbb{R}) \cong \mathbb{R}^*$

② $\Phi : \mathbb{Z} \to \mathbb{Z}_n$

$x \in \mathbb{Z} \mapsto x \bmod n$

$\ker \Phi = \{ x \in \mathbb{Z},\ x \bmod n = 0 \}$

$= n\mathbb{Z}$ (all multiples of n)

Φ is surjective (onto) i.e. $\Phi(\mathbb{Z}) = \mathbb{Z}_n$

$\Rightarrow \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

have seen: Φ hom. $\Rightarrow$ get normal subgroup $\ker \Phi \subset G$

<u>Question:</u> Given a normal subgroup $N \trianglelefteq G$

is there a hom. Φ such that $\ker \Phi = N$?

<u>Answer:</u> Yes

Theorem: Given normal subgroup $N \trianglelefteq G$, we can find a hom.om. Φ with $\ker \Phi = N$

Proof need to construct a hom. Φ

Observation: $N \trianglelefteq G \Rightarrow$ get factor group $G/N = H$

Define map
$$\Phi : G \to G/N$$
$$g \mapsto gN$$

Φ is a homomorphism!

$$\Phi(gh) = ghN \underset{\substack{\uparrow \\ \text{def. of multiplication} \\ \text{in } G/N}}{=} (gN)(hN) = \Phi(g)\Phi(h) \quad \checkmark$$

identity elem. of $G/N = eN = N$

$$\Rightarrow \ker \Phi = \{g \in G, \Phi(g) = N\} = \{g \in G, gN = N\} = \{g \in G, g \in N\} = N \checkmark$$

Chapter 11 Finite Abelian Groups

Goal: Classify all finite abelian groups

Main Result:

① Every finite abelian group is isomorphic to a direct product of cyclic groups

② $|G| = p^a$ $\quad p$ prime

$$\Rightarrow \quad G \cong \mathbb{Z}_{p^{a_1}} \oplus \mathbb{Z}_{p^{a_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{a_r}}$$

where $a_1 + a_2 + \cdots + a_r = a$

Examples:

$a=2$: have 2 possibilities:

$G \cong \mathbb{Z}_{p^2}$

or $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$

$\underline{a=3}$ $\quad G \cong \mathbb{Z}_{p^3}$

or $G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$

or $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$

$\underline{a=4}$ $\quad G \cong \mathbb{Z}_{p^4}$

or $G \cong \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$

or $G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$

or $G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$

or $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$

To make this more precise we say

$(a_1, a_2, \ldots, a_r)$ is a partition of the positive integer a

if $a_i \geq 0$ integer and $a_1 + a_2 + \ldots + a_r = a$

Theorem Abelian groups of order p^a are $\underbrace{\text{classified}}_{\text{up to isomorphism}}$ by $\underbrace{\text{all}}_{\text{possible}}$ partitions $(a_1, a_2 \dots a_r)$ of a.

i.e.

- for every partition $(a_1, a_2 \dots a_r)$ we have the abelian group

$$\mathbb{Z}_{p^{a_1}} \oplus \mathbb{Z}_{p^{a_2}} \oplus \dots \oplus \mathbb{Z}_{p^{a_r}}$$

- groups belonging to different partitions are non-isomorphic.
- any abelian group of order p^a must be isomorphic to one of the groups above.

Def $\mathrm{Par}(a) = \#\{\text{partitions of } a\}$

have sen

$\mathrm{Par}(2) = 2 \quad \{(2), (1,1)\}$

$\mathrm{Par}(3) = 3 \quad \{(3), (2,1), (1,1,1)\}$

$\mathrm{Par}(4) = 5 \quad \{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}$

Question: What about groups of order, say $p^a q^b$, p, q primes?

Answer: $G \cong G_p \oplus G_q$

with $|G_p| = p^a$ $|G_q| = q^b$

$$\left.\begin{array}{l} \Rightarrow \text{ have } Par(a) \text{ possibilities for } G_p \\ \quad\ \text{"} \quad Par(b) \quad \text{"} \qquad\quad \text{"} \quad G_q \end{array}\right\} \text{ have } Par(a)\,Par(b) \text{ possibilities for } G$$

Example: How many abelian groups of order 72, up to isom.?

Answer: $72 = 8 \cdot 9 = 2^3 3^2$

$$\left.\begin{array}{l} \text{have } Par(3) \text{ possibilities for } G_2 \\ \quad\text{"} \quad Par(2) \quad \text{"} \qquad\quad \text{for } G_3 \end{array}\right\} \begin{array}{l} Par(3) \cdot Par(2) \text{ possibilities} \\ = 3 \cdot 2 = \boxed{6} \end{array}$$

$$|G_2| = 8 \Rightarrow G_2 \cong \begin{cases} \mathbb{Z}_8 \\ \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{cases}$$

$|G_3| = 9 = 3^2$

$$G_3 \text{ is isom. to } \begin{cases} \mathbb{Z}_9 \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 \end{cases}$$

General Result

Classify all abelian groups of order n.

- step 1: Find prime factorization of $n = \prod p^{a(p)}$

$$= p_1^{a(p_1)} \, p_2^{a(p_2)} \cdots$$

- step 2: Theorem tells us:

$$\text{If } |G| = n \Rightarrow G \cong G_{p_1} \oplus G_{p_2} \oplus \cdots \oplus G_{p_n}$$

$$|G_{p_i}| = p_i^{a(p_i)}$$

- step 3: have $\mathrm{Par}(a(p_i))$ possibilities for G_{p_i}

Step 4 have $\mathrm{Par}(a_1) \cdot \mathrm{Par}(a_2) \cdot \ldots \cdot \mathrm{Par}(a_r)$

possible groups of order $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$

Example: how many abelian groups of order 105, up to isom.?

Solution: $105 = 3 \cdot 5 \cdot 7$

$$\Rightarrow G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7 \cong \mathbb{Z}_{105}$$

(here all exponents equal to 1)

Recall: For every integer n there exists the cyclic group $\mathbb{Z}_n$ of order n.

Question: Can you find an element in $\mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$ whose order is 105?

Sol. Recall: If $G = G_1 \oplus G_2 \oplus G_3$

and $(g_1, g_2, g_3) \in G$

$$\Rightarrow \operatorname{ord}(g_1, g_2, g_3) = \operatorname{lcm}(\operatorname{ord}(g_1), \operatorname{ord}(g_2), \operatorname{ord}(g_3))$$

can pick $(1,1,1) \in \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$

$$\Rightarrow \operatorname{ord}(1,1,1) = \operatorname{lcm}(\underset{\text{in } \mathbb{Z}_3}{\operatorname{ord}(1)}, \underset{\text{in } \mathbb{Z}_5}{\operatorname{ord}(1)}, \underset{\text{in } \mathbb{Z}_7}{\operatorname{ord}(1)})$$

$$= \operatorname{lcm}(3, 5, 7)$$

$$= 3 \cdot 5 \cdot 7 = 105 \quad \checkmark$$